Suggested Solution to Exercise 3

1. Describe the following curves and express them in the form f(x, y) = 0.

   (a) $x = t^2 + 1,\ y = t - 1 \quad t \in \mathbb{R}$;

   (b) $x = \sin^2 t,\ y = \cos t, \quad t \in \mathbb{R}$;

   (c) $x = t \cos t,\ y = t \sin t, \quad t > 0$.

   Solution. (a) The equation is $x = y^2 + 2y + 2$ which describes a parabola. As y runs from 1 to 0, the curve lies in the fourth quadrant and hits $(2, 0)$ at $t = 0$. Then it goes to $(1, 1)$ as t runs from 0 to 1 in the first quadrant.

   (b) The curve is part of the parabola $x = 1 - y^2$. As $x = \sin^2 t$ is always non-negative, the particle keeps running back and forth between the points $(0, 1)$ and $(-1, 0)$ along an arc of the parabola in the first and fourth quadrants infinitely many times.

   (c) This is the Archimedean spiral $x^2 + y^2 = \arctan y/x$.

2. In plane geometry the ellipse is defined as the loci of the points whose sum of distance to two fixed points is constant. Let these two points be $(c, 0)$ and $(-c, 0), c > 0$ and $2a$ be the sum. Show the loci (x, y) satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 , \quad b^2 = a^2 - c^2 .$$

   Solution. We have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a .$$

   We now rearrange terms to have the desired form. First, taking square of

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} ,$$

   and simplify to get

$$xc - a^2 = -a\sqrt{(x-c)^2 + y^2} .$$

   Taking square again to get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

3. Let $(c, 0)$ and $(-c, 0)$ be given and let H be the set of all points (x, y) whose difference in distances to $(c, 0)$ and $(-c, 0)$ is a constant $2a$.

   (a) Show that H is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

   (b) Show that it admits the parametric equations

$$x = \pm a \cosh t, \quad y = b \sinh t, \quad t \in \mathbb{R} .$$

Solution. (a) From

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=\pm 2a\ ,$$

we get $x^2/a^2-y^2/b^2=1$ as in the case of ellipse. (b) is a direct check.

4. * Study the cycloid

$$x=r(\alpha-\sin\alpha),\quad y=r(1-\cos\alpha),\quad \alpha\in(-\infty,\infty),\ r>0\ .$$

Solution. Look up chapter 11, [Thomas] or google.

5. Consider the Lissajous curve

$$x=A\sin(at+\delta),\quad y=B\sin bt\ ,\quad t\in\mathbb{R}\ ,$$

where A,B,a,b,δ are positive constants. Show that the curve is closed if and only if a/b is a rational number. Here a curve is closed if there exists some T such that $(x(t),y(t))=(x(t+T),y(t+T))$ for all t.

Solution. We could write $x=A\cos\delta\sin at+A\sin\delta\cos at$. As $\sin t$ and $\cos t$ are of period $2n\pi$, $\sin at$ and $\cos at$ are of period $2n\pi/a$ and $\sin bt$ and $\cos bt$ are of period $2m\pi/b$. A common period must be of the form $2n\pi/a=2m\pi/b$, that is, $b/a=m/n\in\mathbb{Q}$.

Note. Google for more.

6. The folium of Descartes in parametric form is given by

$$x=\frac{3at}{1+t^3},\quad y=\frac{3at^2}{1+t^3},\quad a>0\ .$$

(a) Show that it defines a regular curve on $(-\infty,-1)$ and $(-1,\infty)$.

(b) Verify that it is the solution set to

$$x^3+y^3=3axy\ .$$

(c) Sketch its graph.

Solution.

(a) We differentiate x,y to get

$$x'(t)=\frac{3a-6at^3}{(1+t^3)^2}\ ,\quad y'(t)=\frac{6at-3at^3}{(1+t^3)^2}\ .$$

One finds that $(x'(t),y'(t))\neq(0,0)$ for all $t\neq 1$. It defines a regular curve.

(b) When $x=\dfrac{3at}{1+t^3},y=\dfrac{3at^2}{1+t^3}$, we have

$$\begin{aligned}x^3+y^3 &= \frac{27a^3t^3(1+t^3)}{(1+t^3)^3}\\ &= \frac{27a^3t^3}{(1+t^3)^2}\\ &= 3axy\ .\end{aligned}$$

7. Find the velocity, speed and acceleration vectors of the following motions:

   (a) $\mathbf{x}(t) = (t^2, t^3)$,

   (b) $\mathbf{x}(t) = (\cos t, \sin t, e^t)$,

   (c) $\mathbf{r}(t) = (t, 4\tan t, 6t^2 - t^3)$.

   Solution.

   (a) Velocity vector is $\mathbf{x}'(t) = (2t, 3t^2)$.
   Speed is $|\mathbf{x}'(t)| = \sqrt{(2t)^2 + (3t^2)^2} = |t|\sqrt{4 + 9t^2}$.
   Acceleration vector is $\mathbf{x}''(t) = (t, 6t)$.

   (b) Velocity vector is $\mathbf{x}'(t) = (-\sin t, \cos t, e^t)$.
   Speed is $|\mathbf{x}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (e^t)^2} = \sqrt{1 + e^{2t}}$.
   Acceleration vector is $\mathbf{x}''(t) = (-\cos t, -\sin t, e^t)$.

   (c) Velocity vector is $\mathbf{r}'(t) = (1, 4\sec^2 t, 12t - 3t^2)$.
   Speed is $|\mathbf{r}'(t)| = \sqrt{1^2 + (4\sec^2 t)^2 + (12t - 3t^2)^2} = \sqrt{1 + 16\sec^4 t + 144t^2 - 72t^3 + 9t^4}$.
   Acceleration vector is $\mathbf{r}''(t) = (0, 8\sec^2 t \tan t, 12 - 6t)$.

8. Find the position $\mathbf{x}(t)$ of the motion in space when the acceleration and initial data are specified:

   (a)
   $$\mathbf{a}(t) = (6t, -1, 12t^2); \quad \mathbf{x}(0) = (0, 0, 0), \quad \mathbf{v}(0) = (1, 1, 0)\ .$$

   (b)
   $$\mathbf{a}(t) = (\cos t, \sin t, 1); \quad \mathbf{x}(0) = (100, 20, 0), \quad \mathbf{v}(0) = (0, 0, 5)\ .$$

   Solution.

   (a) $\mathbf{a}(t) = (6t, -1, 12t^2)$. Integrating $\mathbf{a}(t)$ to obtain $\mathbf{v}(t)$:

   $$\begin{aligned}\mathbf{v}(t) &= \int_0^t a(\tau)d\tau + v(0)\\ &= (3t^2, -t, 4t^3) + (1, 1, 0)\\ &= (3t^2 + 1, -t + 1, 4t^3).\end{aligned}$$

   Again, integrating $\mathbf{v}(t)$ to obtain $\mathbf{x}(t)$:

   $$\begin{aligned}\mathbf{x}(t) &= \int_0^t \mathbf{v}(\tau)d\tau + \mathbf{x}(0)\\ &= (t + t^3, -\frac{t^2}{2} + t, t^4) + (0, 0, 0)\\ &= (t + t^3, -\frac{t^2}{2} + t, t^4).\end{aligned}$$

(b) $\mathbf{a}(t) = (\cos t, \sin t, 1)$. Integrating $\mathbf{a}(t)$ to get $\mathbf{v}(t)$:

$$\begin{aligned}\mathbf{v}(t) &= \int_0^t \mathbf{a}(\tau)d\tau + \mathbf{v}(0)\\ &= (\sin t, -\cos t + 1, t - 0) + (0, 0, 5)\\ &= (\sin t, -\cos t + 1, t + 5).\end{aligned}$$

Again, Integrating $\mathbf{v}(t)$ to get $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t \mathbf{v}(\tau)d\tau + \mathbf{x}(0)\\ &= (-\cos t + 1, -\sin t + t, \frac{t^2}{2} + 5t - 0) + (100, 20, 0)\\ &= (-\cos t + 101, -\sin t + t + 20, \frac{t^2}{2} + 5t).\end{aligned}$$

9. A particle moves on the unit sphere centered at the origin with constant speed. Show that its velocity is always tangent to the sphere.

Solution. Let $\mathbf{r}(t)$ be its position vector. Since it lies on the unit sphere

$$1 = |\mathbf{r}(t)|^2 = x^2(t) + y^2(t) + z^2(t) \ , \quad \forall t \ ,$$

differentiate both sides with respect to t to yield

$$0 = 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 2\mathbf{r}(t) \cdot \mathbf{v}(t),$$

hence $\mathbf{v}(t)$ is perpendicular to its position $\mathbf{r}(t)$, i.e. $\mathbf{v}(t)$ is tangent to the sphere.

10. A particle moves along a parametric curve with constant speed. Prove that its acceleration is always perpendicular to its velocity.

Solution. Let $\mathbf{r}(t)$ be its position vector with velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$. Since it has constant speed c,

$$c^2 = |\mathbf{r}'(t)|^2 = x'^2(t) + y'^2(t) + z'^2(t).$$

Differentiate both sides with respect to t, we have

$$0 = 2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 2\mathbf{v}(t) \cdot \mathbf{a}(t) \ ,$$

hence $\mathbf{a}(t)$ is perpendicular to its velocity $\mathbf{v}(t)$.

11. Determine the maximum height of a projectile in the plane under the following information: Initial position (0,0), initial speed 160 m/sec, angle of inclination $\pi/6$.

Solution. Let g be the acceleration due to gravity, then $\mathbf{a}(t) = (0, -g)$, $\mathbf{r}(0) = (0, 0)$, $\mathbf{v}(0) = 160\Big(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\Big) = 160\Big(\frac{\sqrt{3}}{2}, \frac{1}{2}\Big)$

Integrating $\mathbf{a}(t)$ to get $\mathbf{v}(t)$:

$$\begin{aligned}\mathbf{v}(t) &= \int_0^t \mathbf{a}(\tau)d\tau + \mathbf{v}(0)\\ &= (0,-gt) + 160(\frac{\sqrt{3}}{2}, \frac{1}{2})\\ &= (80, 80\sqrt{3} - gt)\ .\end{aligned}$$

Again, integrating $\mathbf{v}(t)$ to get $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t \mathbf{v}(\tau)d\tau + \mathbf{x}(0)\\ &= (80t, 80\sqrt{3}t - \frac{1}{2}gt^2)\ .\end{aligned}$$

The height function is given by $h(t) = 80\sqrt{3}t - \frac{1}{2}gt^2$. To find its maximum, we set $h'(t) = 0$ to find the maximum point $t_0 = \frac{80\sqrt{3}}{g}$.

Therefore, the maximum height is $h(t_0) = 80\sqrt{3} \cdot \frac{80\sqrt{3}}{g} - \frac{1}{2}g\left(\frac{80\sqrt{3}}{g}\right)^2 = \frac{(80\sqrt{3})^2}{2g}$ (in m).

12. A projectile is fired horizontally from a 1 km-cliff to reach 2 km from the base of the cliff. What should be the initial velocity ?

    Solution. Now we have $\mathbf{a}(t) = (0,-g), \mathbf{r}(0) = (0,1), \mathbf{v}(0) = (c,0)$, where $c \in \mathbb{R}$ is the initial (horizonal) velocity.

    Integrating $\mathbf{a}(t)$ to get $\mathbf{v}(t)$:

$$\begin{aligned}\mathbf{v}(t) &= \int_0^t \mathbf{a}(\tau)d\tau + \mathbf{v}(0)\\ &= (0,-gt) + (c,0)\\ &= (c,-gt).\end{aligned}$$

Again, integrating $v(t)$ to get $x(t)$:

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t v(\tau)d\tau + x(0)\\ &= (ct, -\frac{1}{2}gt^2) + (0,1)\\ &= (ct, -\frac{1}{2}gt^2 + 1).\end{aligned}$$

From $\mathbf{x}(t_0) = (2,0)$ we find $t_0 = \sqrt{2/g}$ and $ct_0 = 2$. Thus $c = \sqrt{2g}$.

13. Let $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ be two differentiable curves from some interval to $\mathbb{R}^n$. Establish the following product rules:

    (a)
    $$\frac{d}{dt}\boldsymbol{\gamma}\cdot\boldsymbol{\eta} = \boldsymbol{\gamma}'\cdot\boldsymbol{\eta} + \boldsymbol{\gamma}\cdot\boldsymbol{\eta}' \ .$$

    (b)
    $$\frac{d}{dt}\boldsymbol{\gamma}\times\boldsymbol{\eta} = \boldsymbol{\gamma}'\times\boldsymbol{\eta} + \boldsymbol{\gamma}\times\boldsymbol{\eta}',$$

    when $n = 3$.

    Solution

    (a) Let $\boldsymbol{\gamma}(t) = (\gamma_1(t), ..., \gamma_n(t))$ and $\boldsymbol{\eta}(t) = (\eta_1(t), ..., \eta_n(t))$. Then $\boldsymbol{\gamma}(t)\cdot\boldsymbol{\eta}(t) = \gamma_1(t)\eta_1(t) + ... + \gamma_n(t)\eta_n(t)$. Therefore,

    $$\begin{aligned}\frac{d}{dt}(\boldsymbol{\gamma}\cdot\boldsymbol{\eta}) &= \frac{d}{dt}(\gamma_1(t)\eta_1(t) + ... + \gamma_n(t)\eta_n(t))\\ &= (\gamma_1'(t)\eta_1(t) + \gamma_1(t)\eta_1'(t)) + ... + (\gamma_n'(t)\eta_n(t) + \gamma_n(t)\eta_n'(t))\\ &= (\gamma_1'(t)\eta_1(t) + \gamma_2'(t)\eta_2(t) + ... + \gamma_n'(t)\eta_n(t)) + (\gamma_1(t)\eta_1'(t) + \gamma_2(t)\eta_2'(t) + ... + \gamma_n(t)\eta_n'(t))\\ &= \boldsymbol{\gamma}'(t)\cdot\boldsymbol{\eta}(t) + \boldsymbol{\gamma}(t)\cdot\boldsymbol{\eta}'(t).\end{aligned}$$

    (b) Let $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t), \eta_3(t))$. Then $\boldsymbol{\gamma}(t)\times\boldsymbol{\eta}(t) = (\gamma_2(t)\eta_3(t) - \gamma_3(t)\eta_2(t), \gamma_3(t)\eta_1(t) - \gamma_1(t)\eta_3(t), \gamma_1(t)\eta_2(t) - \gamma_2(t)\eta_1(t))$. Therefore,

    $$\begin{aligned}\frac{d}{dt}(\boldsymbol{\gamma}\times\boldsymbol{\eta}) &= \frac{d}{dt}(\gamma_2(t)\eta_3(t) - \gamma_3(t)\eta_2(t), \gamma_3(t)\eta_1(t) - \gamma_1(t)\eta_3(t), \gamma_1(t)\eta_2(t) - \gamma_2(t)\eta_1(t))\\ &= ((\gamma_2'(t)\eta_3(t) + \gamma_2(t)\eta_3'(t)) - (\gamma_3'(t)\eta_2(t) + \gamma_3(t)\eta_2'(t)),\\ &\quad (\gamma_3'(t)\eta_1(t) + \gamma_3(t)\eta_1'(t)) - (\gamma_1'(t)\eta_3(t) + \gamma_1(t)\eta_3'(t)),\\ &\quad (\gamma_1'(t)\eta_2(t) + \gamma_1(t)\eta_2'(t)) - (\gamma_2'(t)\eta_1(t) + \gamma_2(t)\eta_1'(t))\\ &= (\gamma_2'(t)\eta_3(t) - \gamma_3'(t)\eta_2(t), \gamma_3'(t)\eta_1(t) - \gamma_1'(t)\eta_3(t), \gamma_1'(t)\eta_2(t) - \gamma_2'(t)\eta_1(t))\\ &\quad + (\gamma_2(t)\eta_3'(t) - \gamma_3(t)\eta_2'(t), \gamma_3(t)\eta_1'(t) - \gamma_1(t)\eta_3'(t), \gamma_1(t)\eta_2'(t) - \gamma_2(t)\eta_1'(t))\\ &= \boldsymbol{\gamma}'\times\boldsymbol{\eta} + \boldsymbol{\gamma}\times\boldsymbol{\eta}' \ .\end{aligned}$$

14. Let $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ be two regular curves on some interval and $\boldsymbol{\gamma}(t_1) = \boldsymbol{\eta}(t_2)$. Define the angle between these two curves at this point of intersection to the angle $\theta \in [0, \pi/2]$ between the two tangent lines passing this point. Show that

    $$\cos\theta = \frac{|\gamma_1'(t_1)\eta_1'(t_2) + \gamma_2'(t_1)\boldsymbol{\eta}'(t_2)|}{\sqrt{(\gamma_1'^2(t_1) + \eta_1'^2(t_2))(\gamma_2'^2(t_1) + \eta_2'^2(t_2))}} \ .$$

    Solution. Let $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$ and $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t))$. The corresponding tangent vectors at $\boldsymbol{\gamma}(t_1) = \boldsymbol{\eta}(t_2)$ is given by $\boldsymbol{\gamma}'(t_1) = (\gamma_1'(t_1), \gamma_2'(t_1))$ and $\boldsymbol{\eta}'(t_2) = (\eta_1'(t_2), \eta_2'(t_2))$. The angle is then given by

$$\cos\theta = \frac{|\boldsymbol{\gamma}'(t)\cdot\boldsymbol{\eta}'(t)|}{|\boldsymbol{\gamma}'(t)||\boldsymbol{\eta}'(t)|}$$
$$= \frac{|\gamma_1'(t_1)\eta_1'(t_2)+\gamma_2'(t_1)\eta_2'(t_2)|}{\sqrt{(\gamma_1'^2(t_1)+\eta_1'^2(t_2))(\gamma_2'^2(t_1)+\eta_2'^2(t_2))}}$$

15. * Let γ and η be two regular curves in $\mathbb{R}^n$. Suppose $P=\boldsymbol{\gamma}(t_0)$ and $Q=\boldsymbol{\eta}(s_0)$ are points realizing the (minimal) distance between these two curves. Show that

$$\overline{PQ}\cdot\boldsymbol{\gamma}'(t_0)=\overline{PQ}\cdot\boldsymbol{\eta}'(s_0)=0.$$

Solution. Let
$$f(t)=|\boldsymbol{\gamma}(t)-\boldsymbol{\eta}(s_0)|^2.$$
It attains its minimum at $t=t_0$. Therefore,
$$f'(t_0)=2\overline{PQ}\cdot\boldsymbol{\gamma}'(t_0)=0\ .$$
Similarly, we get the other relation by differentiating
$$g(s)=|\boldsymbol{\gamma}(t_0)-\boldsymbol{\eta}(s)|^2.$$

16. The circle $x^2+y^2=1$ can be described by the graphs of two functions, $f_1(x)=\sqrt{1-x^2}$ and $f_2(x)=1\sqrt{1-x^2}$ over $[-1,1]$. However, both functions are not differentiable at $x=\pm1$. Can you described the circle in terms of four differentiable functions over some intervals of the x- or y-axis ?

Solution. We define the following four functions: $f_1(x)=\sqrt{1-x^2}$, $f_2(x)=-\sqrt{1-x^2}$ on $x\in(-2/3,2/3)$; $f_3(y)=-\sqrt{1-y^2}$, $f_4(x)=\sqrt{1-y^2}$ on $y\in(-2/3,2/3)$. Then they are differentiable on their respective domains, and their graphs completely describe the circle.

17. Write down the polar equation for a straight line not passing through the origin.

Solution. Equation of such straight line in rectangular coordinates is given by

$$ax+by=c$$

where $c\neq0$. Apply the polar coordinate change of variables $x=\rho\cos\theta, y=\rho\sin\theta$, we have
$$a(\rho\cos\theta)+b(\rho\sin\theta)=c$$
and hence
$$\rho=\frac{c}{a\cos\theta+b\sin\theta}$$
which is the polar equation of the straight line.

Note. Assume that $a,b>0$, and set $\tan\theta_0=-a/b, \theta_0\in(-\pi/2,0)$. Then $\theta\in(-\theta_0,\pi-\theta_0)$ is the domain of the parameter.

18. Show that the polar equation for the circle centered at $(a, 0)$ with radius a, where $a > 0$, is given by

$$\rho(\theta) = 2a\cos\theta, \quad \theta \in (-\pi/2, \pi/2] \ .$$

Solution Equation of such a circle in rectangular coordinates is given by

$$(x-a)^2 + y^2 = a^2$$

which simplifies to

$$x^2 + y^2 = 2ax$$

Apply the polar coordinate change of variables $x = \rho\cos\theta, y = \rho\sin\theta$, we have

$$(\rho\cos\theta)^2 + (\rho\sin\theta)^2 = 2a\rho\cos\theta$$

which simplifies to

$$\rho^2 = 2a\rho\cos\theta$$

Since $\rho > 0$, it further simplifies to

$$\rho = 2a\cos\theta$$

which is the polar equation of the circle.

19. Sketch the graphs of the following polar equations and convert them to the form $f(x, y) = 0$.

(a) The 3-leave Rhodonea

$$\rho = a\cos 3\theta \ , \quad a > 0.$$

(b) The astroid

$$x = 4a\cos^3 t, \quad y = 4a\sin^3 t, \quad a > 0 \ .$$

(c) The logarithmic spiral

$$r = e^{b\theta} \ , \quad b > 0 \ .$$

Solution. (a) $\cos 3\theta$ is of period $2\pi/3$. It is completely described in any interval of length $2\pi/3$. Let us take it to be $[-\pi/6, 5\pi/6]$. When $\theta \in [-\pi/6, \pi/6]$, the curve forms a leaf in the first and fourth quadrants symmetric with respect to the x-axis. When $\theta \in [\pi/6, \pi/2]$, r is non-positive. It forms an identical leaf in the third quadrant. Finally, another leaf lies in the second quadrant when $\theta \in [\pi/2, 5\pi/6]$. Using $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, this curve satisfies

$$(x^2 + y^2)^2 = 4ax^3 - 3ax(x^2 + y^2) \ .$$

(b) This curve is of period 2π. It is given by $x^{2/3} + y^{2/3} = a^{2/3}$, or

$$(x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0 \ .$$

It has four cusps at $(\pm 4a, 0)$ and $(0, \pm 4a)$.

(c) The logarithmic spiral satisfies

$$x^2 + y^2 = e^{2b\arctan y/x} \ .$$

20. Consider the one-dimensional motion described by

$$m\frac{d^2x}{dt^2} = -kx \ ,$$

where $k > 0$ and $x(t)$ is the location of the particle at time t.

(a) Show that $x'^2 + \frac{k}{m}x^2$ is constant in time.

(b) Using (a) to show the motion must be of the form

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}(t - t_0)\right) , \qquad \text{for some } A, t_0 \in \mathbb{R} \ .$$

(c) Show that the general solution of the differential equation

$$m\frac{d^2x}{dt^2} = -kx + b \ , \quad b \in \mathbb{R},$$

is given by

$$x(t) = \frac{b}{k} + A\cos\left(\sqrt{\frac{k}{m}}(t - t_0)\right) .$$

Solution.

(a) We compute

$$\begin{aligned}\frac{d}{dt}(x'^2 + \frac{k}{m}x^2) &= 2x'x'' + 2\frac{k}{m}xx' \\ &= 2x'(x'' + \frac{k}{m}x) \\ &= 0.\end{aligned}$$

Therefore, $x'^2 + \frac{k}{m}x^2$ is constant in time.

(b) By (a), $x'^2 + \frac{k}{m}x^2 = c_0$ for some $c_0 \geq 0$. When $c_0 = 0$, $x(t) = 0$ for all $t \in \mathbb{R}$). When $c_0 > 0$: then

$$x'(t) = \pm\sqrt{c_0 - \frac{k}{m}x^2}.$$

By integrating

$$\frac{dx}{\sqrt{c_0 - \frac{k}{m}x^2}} = \pm dt \ ,$$

we have

$$x(t) = \pm\frac{k}{m}\frac{1}{c_0}\cos\left(\sqrt{\frac{k}{m}}(t - t_0)\right) \ .$$

(c) Let $y = x - \frac{b}{k}$. The differential equation becomes

$$\begin{aligned}m\frac{d^2y}{dt^2} &= m\frac{d^2x}{dt^2} \\ &= -kx + b \\ &= -k(y + \frac{b}{k}) \\ &= -ky \ .\end{aligned}$$

By (b),

$$y(t) = A\cos\left(\sqrt{\frac{k}{m}}(t-t_0)\right) .$$

Therefore,

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}(t-t_0)\right) + \frac{b}{k} .$$

21. * Find the length of the following parametric curves:

(a)

$$\mathbf{r}(t) = (3\sin 2t, 3\cos 2t, 8t), \quad t \in [0, \pi].$$

(b)

$$\mathbf{x}(t) = \left(t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{\sqrt{3}}\right), \quad t \in [0, 1] .$$

(c)

$$\boldsymbol{\gamma}(t) = (2e^t, e^{-t}, 2t), \quad t \in [0, 1] .$$

Solution

(a) $\mathbf{r}'(t) = (6\cos 2t, -6\sin 2t, 8)$. Therefore, $|r'(t)|^2 = (6\cos 2t)^2 + (-6\sin 2t)^2 + 8^2 = 100$, and hence $|\mathbf{r}'(t)| = 10$ for all $t \in [0, \pi]$.
Therefore, its length is given by

$$L = \int_0^{\pi} |\mathbf{r}'(t)| dt = 10\pi.$$

(b) $\mathbf{x}'(t) = (1, \sqrt{2}t, \sqrt{3}t^2)$. Therefore, $|\mathbf{x}'(t)|^2 = (1)^2 + (\sqrt{2}t)^2 + (\sqrt{3}t^2)^2 = 1 + 2t^2 + 3t^4$, and hence $|\mathbf{x}'(t)| = \sqrt{1+2t^2+3t^4}$ for all $t \in [0, 1]$.
The length is given by

$$\begin{aligned} L &= \int_0^1 |\mathbf{x}'(t)| dt \\ &= \int_0^1 \sqrt{1+2t^2+3t^4}\ dt. \end{aligned}$$

No need to go further.

(c) $\boldsymbol{\gamma}'(t) = (2e^t, -e^{-t}, 2)$. Therefore, $|\boldsymbol{\gamma}'(t)|^2 = (2e^t)^2 + (-e^{-t})^2 + (2)^2 = (2e^t + e^{-t})^2$, and hence $|\boldsymbol{\gamma}'(t)| = 2e^t + e^{-t}$ for all $t \in [0, 1]$.
Therefore, its length is given by

$$\begin{aligned} L &= \int_0^1 |\boldsymbol{\gamma}'(t)| dt \\ &= \int_0^1 (2e^t + e^{-t}) dt \\ &= (2e - e^{-1}) - 1. \end{aligned}$$